The Mermin - Wagner Theorem

We know that in 1D the Ising model is ordered only at $T=0$.

What about the $O(n)$ model, namely a model with vectorial fields with n components?

The Hamiltonian is, as we know

$$H = \int d\vec{x} \left[c(\nabla\vec{\phi})^2 + r\vec{\phi}^2 + u(\vec{\phi}^2)^2 \right]$$

Let's say that the system is "almost" ordered, that is $|\vec{\phi}| = |\vec{\phi}_0| = \phi_0$ and we write

$$\vec{\phi} = \vec{\phi}_0 + \underbrace{\delta\vec{\phi}}_{\text{very small}} \quad \text{with } \vec{\phi}_0 \text{ uniform}$$

Then the terms of the Hamiltonian are

$$\vec{\phi}^2 = (\vec{\phi}_0 + \delta\vec{\phi})^2 = \phi_0^2 + 2\vec{\phi}_0 \cdot \delta\vec{\phi} + \delta\vec{\phi}^2$$

and

$$(\vec{\phi}^2)^2 = \phi_0^4 + 4(\vec{\phi}_0 \cdot \delta\vec{\phi})^2 + (\delta\vec{\phi}^2)^2 +$$

$$+ 4\phi_0^2\, \vec{\phi}_0 \cdot \delta\vec{\phi} + 2\phi_0^2\, \delta\vec{\phi}^2 + 4\, \vec{\phi}_0 \cdot \delta\vec{\phi}(\delta\vec{\phi}^2)$$

Let's decompose $\delta\vec{\phi}$:

$$\underbrace{\delta\vec{\phi}}_{} = \delta\phi_{\parallel} + \delta\vec{\phi}_{\perp}$$

orthogonal to $\vec{\phi}_0$

parallel to $\vec{\phi}_0$

then

$$\vec{\phi}_0 \cdot \delta\vec{\phi} = \phi_0\, \delta\phi_{\parallel}$$

$$\delta\vec{\phi}^2 = \delta\phi_{\parallel}^2 + (\delta\phi_{\perp})^2$$

lets compute now $\quad r\vec{\phi}^2 + u(\vec{\phi}^2)^2$:

$$r\phi_0^2 + 2r\phi_0\,\delta\phi_{\parallel} + r\,\delta\phi_{\parallel}^2 + r\,\delta\phi_{\perp}^2 +$$

$$+ u\phi_0^4 + 4u\phi_0^2\,\delta\phi_{\parallel}^2 + u\,\delta\phi_{\parallel}^4 + u\,\delta\vec{\phi}_{\perp}^4 + 2u\,\delta\phi_{\parallel}^2\,\delta\vec{\phi}_{\perp}^2 + 4u\phi_0^3\,\delta\phi_{\parallel} +$$

$$+ 2u\phi_0^2(\delta\phi_{\parallel}^2 + \delta\vec{\phi}_{\perp}^2) + 4u\phi_0\,\delta\phi_{\parallel}(\delta\phi_{\parallel}^2 + \delta\vec{\phi}_{\perp}^2) =$$

$$= (r\phi_0^2 + u\phi_0^4) + (2r\phi_0 + 4u\phi_0^3)\,\delta\phi_{\parallel} + (r + 2u\phi_0^2)\,\delta\phi_{\parallel}^2 +$$

$$+ 4u\phi_0^2\,\delta\phi_{\parallel}^2 + \underbrace{o(\delta\phi_{\parallel}^2)}_{} +$$

small because $\delta\phi$ is very small

$$+ (r + 2u\phi_0^2)\,\delta\vec{\phi}_{\perp}^2 + o(\delta\vec{\phi}_{\perp}^2)$$

again small

Now let's focus on ϕ_0: we have said that it is the equilibrium value of the field once the direction is fixed:

$$\frac{d}{d\phi}\left(r\phi^2 + u\phi^4\right)\Big|_{\phi_0} = 2r\phi_0 + 4u\phi_0^3 = 0$$

$$2\phi_0\left(r + 2u\phi_0^2\right) = 0$$

we have assumed no fluctuations: $\nabla\vec{\phi} = 0$

Then we see that the Hamiltonian, in the small $\delta\vec{\phi}$ assumption, reduces to

$$H = \int d\vec{x}\left[r(\nabla\delta\vec{\phi}_\perp)^2 + r(\nabla\delta\phi_\parallel)^2 + 4u\phi_0^2\,\delta\phi_\parallel^2\right] =$$

$$= H_\perp + H_\parallel$$

with
$$H_\perp = \int d\vec{x}\; r(\nabla\delta\vec{\phi}_\perp)^2$$
$$H_\parallel = \int d\vec{x}\left[r(\nabla\delta\phi_\parallel)^2 + 4u\phi_0^2\,\delta\phi_\parallel^2\right]$$

$H_\parallel$ is essentially the one for a field with a minimum in $\delta\phi_\parallel = 0$ because $4u\phi_0^2$ is positive.

Fluctuations in $\delta\phi_\parallel$ are thus small, as postulated.

What about $H_\perp$ and $\delta\vec{\phi}_\perp$?

Let's compute

$$\langle \vec{\phi}^2(\vec{r}) \rangle = \langle (\vec{\Phi}_0 + \delta\vec{\phi})^2 \rangle =$$

$$= \phi_0^2 + 2\phi_0 \langle \delta\phi_\parallel \rangle + \langle \delta\phi_\parallel^2 \rangle + \langle \delta\vec{\phi}_\perp^2 \rangle$$

And consider

$\langle \delta\phi_\parallel \rangle = 0$ because it is computed with a quadratic weight in $\delta\phi_\parallel$ $\Rightarrow$ average of an odd function with an even weight.

$\langle \delta\phi_\parallel^2 \rangle$ we said is small (can be computed but not necessary here)

What about $\langle \delta\vec{\phi}_\perp^2 \rangle$?

$\delta\vec{\phi}_\perp$ has $n-1$ components and their weight is isotropic $\Rightarrow$ we can assume they are all giving the same contribution.

Thus

$$< \delta \vec{\phi}_\perp(\vec{r})^2 > = (n-1) < \delta\phi_\perp^2(\vec{r}) >$$

Let's write the $H_\perp$ in Fourier space

$$H_\perp = \int d\vec{x} \left[r (\nabla \delta\vec{\phi}_\perp)^2 \right] = (n-1) \int d\vec{x} \left[r (\nabla \delta\phi_\perp)^2 \right] =$$

$$= (n-1) \int d\vec{x}\, r \int d\vec{k}\, d\vec{k}' \frac{1}{(2\pi)^{2d}} (i\vec{k})\cdot(i\vec{k}')\, \delta\tilde{\phi}_\perp(\vec{k})\, \delta\tilde{\phi}_\perp(\vec{k}') e^{i(\vec{k}+\vec{k}')\vec{x}} =$$

$$= \int d\vec{k} \frac{(n-1)r}{(2\pi)^d} k^2 \delta\tilde{\phi}_\perp(\vec{k})\, \delta\tilde{\phi}_\perp^*(\vec{k})$$

The Fourier expression of $\delta\phi_\perp^2(\vec{r})$ is

$$\delta\phi_\perp^2(\vec{r}) = \int \frac{d\vec{k}\, d\vec{k}'}{(2\pi)^{2d}} \delta\tilde{\phi}_\perp(\vec{k})\, \delta\tilde{\phi}_\perp(\vec{k}')\, e^{i(\vec{k}+\vec{k}')\cdot\vec{r}}$$

Then

$$< \delta\phi_\perp^2(\vec{r}) > = \int \frac{d\vec{k}\, d\vec{k}'}{(2\pi)^{2d}} < \delta\tilde{\phi}_\perp(\vec{k}) \underbrace{\delta\tilde{\phi}_\perp^*(-\vec{k}')}_{} > e^{i(\vec{k}+\vec{k}')\cdot\vec{r}}$$

$$\delta\tilde{\phi}_\perp^*(-\vec{k}') = \delta\tilde{\phi}_\perp(\vec{k}')$$

The Hamiltonian $H_\perp$ is diagonal in $\vec{k}$, which means that if $\vec{k} \neq -\vec{k}'$ then

$$\langle \delta\tilde{\phi}(\vec{k})\, \delta\tilde{\phi}^*(-\vec{k}') \rangle = \langle \delta\tilde{\phi}(\vec{k}) \rangle \langle \delta\tilde{\phi}^*(-\vec{k}') \rangle = 0$$

$$\Rightarrow \quad \langle \delta\tilde{\phi}(\vec{k})\, \delta\tilde{\phi}^*(-\vec{k}') \rangle \propto \delta(\vec{k}+\vec{k}') \langle \delta\tilde{\phi}(\vec{k})\, \delta\tilde{\phi}^*(\vec{k}) \rangle$$

Then

$$\langle \vec{\phi}^2(\vec{r}) \rangle \propto \phi_0^2 + \text{const}\cdot(n-1)\int d\vec{k}\, \underbrace{\frac{1}{k^2}}_{\langle \delta\tilde{\phi}_\perp(\vec{k})\, \delta\tilde{\phi}_\perp^*(\vec{k}) \rangle} =$$

$$= \phi_0^2 + \text{const}\cdot(n-1)\,\Omega_d \int_{\frac{2\pi}{L}}^{\frac{2\pi}{a}} k^{d-1}\frac{1}{k^2}\, dk =$$

angular integral in d-dimensions

a = minimal scale
L = size of the system

$$= \phi_0^2 + \text{const}\cdot(n-1)\,\Omega_d \underbrace{\int_0^{\frac{2\pi}{a}} k^{d-3}\, dk}_{}$$

large size limit

This integral diverges at the origin if $d \leq 2$

also the equal sign!

This means that the hypothesis the fluctuations were small is inconsistent.

Stated otherwise : a system with continuous symmetry, $n \geq 2$, cannot be ordered in $d \leq 2$ at finite temperature.

This is the Mermin - Wagner theorem.

There is a transition for $n = 2$ in $d = 2$, actually, but it is a "topological one" (the Kosterlitz - Thouless transition).